

**UNITARIZABLE REPRESENTATIONS OF THE DEFORMED
PARA-BOSE SUPERALGEBRA $U_q[\mathfrak{osp}(1/2)]$ AT ROOTS OF 1**

Short title: **UNITARIZABLE ROOT OF 1 IRREPS OF $U_q[\mathfrak{osp}(1/2)]$**

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Abstract. The unitarizable irreps of the deformed para-Bose superalgebra pB_q , which is isomorphic to $U_q[osp(1/2)]$, are classified at q being root of 1. New finite-dimensional irreps of $U_q[osp(1/2)]$ are found. Explicit expressions for the matrix elements are written down.

1. Introduction

In the present paper we study unitarizable root of unity representations of the Hopf algebra $pB_q(1) \equiv pB_q$, introduced in [1]. It is generated (essentially) by one pair of deformed para-Bose operators a^\pm . The irreps of pB_q at generic values of q are infinite-dimensional and are realized in deformed para-Bose Fock spaces $F(p)$, $p \in \mathbf{C}$ [1]. The multimode Hopf algebra $pB_q(n)$, corresponding to n pairs of deformed pB operators $a_1^\pm, a_2^\pm, \dots, a_n^\pm$ was defined in [2-5]. The case of any number of deformed para-Fermi operators was worked out in [6].

So far various deformations of para-Bose and para-Fermi statistics were considered from different points of view [7-22]. Some of them are not related to any Hopf algebra structure. A guiding principle of the approaches in [1-5], which we follow, is to preserve, similar to the nondeformed case [23], the identification of $pB_q(n)$ with $U_q[osp(1/2n)]$: $pB_q(n)$ is an associative superalgebra isomorphic (as a Hopf algebra) to the deformed universal enveloping algebra $U_q[osp(1/2n)]$ of the orthosymplectic Lie superalgebra $osp(1/2n)$.

The Hopf algebra structure of $pB_q(n)$ has an important advantage: using the comultiplication, one can define new representations of the deformed operators (and hence of $U_q[osp(1/2n)]$) in any tensor product of representation spaces. In particular one can use the Fock space of n pairs of commuting deformed Bose operators [24-27], since they give a representation of $U_q[osp(1/2n)]$ [28]. Even in the nondeformed case the only effective technique for constructing representations of parabosons or of $osp(1/2n)$ (for large n) is through tensor products of bosonic Fock spaces (see [29] for more discussions in this respect).

The definition of $U_q[osp(1/2n)]$ in terms of its Chevalley generators is well known [30-35]. Although for $n > 1$ the deformed pB operators $a_1^\pm, a_2^\pm, \dots, a_n^\pm$ are very different from the Chevalley generators, the relations determining $U_q[osp(1/2n)]$ through $a_1^\pm, a_2^\pm, \dots, a_n^\pm$ are not more involved [4,5]. At $n = 1$, namely in the case we consider, a^\pm are proportional to the Chevalley generators of $pB_q = U_q[osp(1/2)]$.

The finite-dimensional irreps of $U_q[osp(1/2)]$ at generic q were constructed in [36, 37]. Some root of unity highest weight irreps were also obtained in [37]; both highest weight and cyclic representations were studied in detail in [38-40].

Our carrier representation spaces $F(p)$, $p \in \mathbf{C}$ will be deformed Fock spaces [1], which are in fact the Verma modules used in [39]. At root of unity cases each such space is no more irreducible; it contains infinitely many invariant subspaces. The irreps are realized in appropriate factor spaces of $F(p)$ with the vacuum being the highest weight vector.

To our best knowledge the root of 1 irreps of $U_q[osp(1/2)]$ obtained in Section 3.2 and those labeled with an integer p [Sections 3.1a, 3.1b, 3.3] were not described in the literature so far. Our other main result is the classification of the unitarizable Fock irreps of pB_q (= unitarizable Verma representations of $U_q[osp(1/2)]$) at roots of 1 (see (4.7)). We write down explicit expressions for the transformation of the basis under the action of the deformed pB generators.

The reason to pay a special attention to the unitarizable representations stems from physical considerations. In all known to us applications of deformed parastatistics [7-22] it is assumed that

the Hermitian conjugate $(a^-)^\dagger$ of the annihilation operator a^- equals to the creation operator a^+ ,

$$(a^-)^\dagger = a^+. \quad (1.1)$$

In case of deformed para-oscillators [9, 15], for instance, or more generally in any deformed quantum mechanics (see for instance [41] and the references therein) the unitarity condition (1.1) is equivalent (as in the canonical case) to the requirement the position and the momentum operators to be selfadjoint operators.

The paper is organized as follows. In Sec. 2 we recall the definition of the deformed para-Bose algebra and its Fock representations at generic q . Sec. 3 is devoted to a detailed study of the root of 1 irreps. The indecomposable representations both finite-dimensional and infinite-dimensional are also mentioned. The unitarizable representations are classified in Sec. 4. Sec. 5 contains some concluding remarks.

Throughout we use the following abbreviations and notation:

- \mathbf{C} — all complex numbers
- \mathbf{Z} — all integers
- \mathbf{Z}_+ — all nonnegative integers
- $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$ — the ring of all integers modulo 2
- $[A, B] = AB - BA, \quad \{A, B\} = AB + BA.$

2. The para-Bose Hopf algebra pB_q and its Fock representations

To begin with we summarize some of the results from [1], changing slightly the notation.

Definition 1. The para-Bose algebra pB_q , $q \in \mathbf{C} \setminus \{0, \pm 1\}$, is the associative superalgebra over \mathbf{C} with unit 1 defined by the following generators and relations

$$\text{Generators : } a^\pm, K^{\pm 1} \quad (2.1)$$

$$\text{Relations : } KK^{-1} = K^{-1}K = 1 \quad Ka^\pm = q^{\pm 2}a^\pm K \quad \{a^+, a^-\} = \frac{K - K^{-1}}{q - q^{-1}} \quad (2.2)$$

$$\mathbf{Z}_2 - \text{grading : } \deg(K^{\pm 1}) = \bar{0} \quad \deg(a^\pm) = \bar{1}. \quad (2.3)$$

The creation operator a^+ (resp. the annihilation operator a^-) is the negative (resp. the positive) root vector of $pB_q = U_q[osp(1/2)]$.

Setting $K = q^H$ with $q = e^\eta$, $\eta \in \mathbf{C}$, one recovers as $q \rightarrow 1$ ($\eta \rightarrow 0$) the defining relations of the nondeformed para-Bose operators [42] ($\xi, \eta, \epsilon = \pm$ or ± 1):

$$[\{a^\xi, a^\eta\}, a^\epsilon] = (\epsilon - \eta)a^\xi + (\epsilon - \xi)a^\eta \quad (2.4)$$

with $H = \{a^+, a^-\}$.

It was already shown in [1] how pB_q can be endowed with a comultiplication, a counity and an antipode; here we shall be not concerned with this additional structure.

The (deformed) Fock space $F(p)$ is defined for any complex number p , $p \in \mathbf{C}$, postulating that $F(p)$ contains a vacuum vector $|p; 0\rangle$, namely

$$a^-|p; 0\rangle = 0 \quad (2.5)$$

so that

$$K|p; 0\rangle = q^p|p; 0\rangle \quad (\Leftrightarrow H|p; 0\rangle = p|p; 0\rangle). \quad (2.6)$$

At the limit $q \rightarrow 1$ the above definition of $F(p)$ reduces to the usual one ($a^-|p; 0\rangle = 0$, $a^-a^+|p; 0\rangle = p|p; 0\rangle$), where p is the order of the parastatistics [42].

$F(p)$ is an infinite-dimensional linear space with a basis

$$|p; n\rangle = (a^+)^n|p; 0\rangle \quad n \in \mathbf{Z}_+ \quad (2.7)$$

and a highest weight vector $|p; 0\rangle$.

Setting

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \{n\} = \frac{q^n + q^{-n}}{q + q^{-1}} \quad (2.8)$$

one can write the transformation of the basis as follows:

$$K|p; n\rangle = q^{2n+p}|p; n\rangle \quad (2.9a)$$

$$a^+|p; n\rangle = |p; n+1\rangle \quad (2.9b)$$

$$a^-|p; n\rangle = [n]\{n+p-1\}|p; n-1\rangle \quad \text{for } n = \text{even number} \quad (2.9c)$$

$$a^-|p; n\rangle = [n+p-1]\{n\}|p; n-1\rangle \quad \text{for } n = \text{odd number}. \quad (2.9d)$$

At generic q each Fock space $F(p)$, $p \in \mathbf{C}$, is an infinite-dimensional simple (=irreducible) pB_q module [1].

3. Root of unity representations

If q is a root of 1 the pB_q module $F(p)$ may no longer be irreducible. More precisely,

Proposition 1. The Fock space $F(p)$ is indecomposable if and only if $q = e^{i\frac{\pi}{2}\frac{m}{k}}$ for every $m, k \in \mathbf{Z}$ such that $q \notin \{\pm 1, \pm i\}$, i.e., $m \neq 0 \pmod{k}$.

Proof. We exclude from consideration $q \in \{\pm 1, \pm i\}$ since at those values of q the expressions (2.9) are not defined (at $q = \pm 1$ also pB_q is undefined).

If $q \neq e^{i\frac{\pi}{2}\frac{m}{k}}$ the coefficients in front of $|p; n\rangle$ in (2.9c,d) vanish only for $n = 0$. Hence the only singular vector is the vacuum, i.e., $F(p)$ is an irreducible module. If $q = e^{i\frac{\pi}{2}\frac{m}{k}}$ then, for instance, $|p; 2k\rangle$ is a singular vector, $a^-|p; 2k\rangle = 0$. Therefore the proper subspace of $F(p)$, spanned on $|p; n\rangle$, $n \geq 2k$ is an invariant subspace. Then eq. (2.9b) yields that the representation is indecomposable. This completes the proof.

The algebras pB_q corresponding to all possible values of m and k contain several isomorphic copies. Clearly we can always assume that $k > 0$ and that m and k are co-prime, i.e., $\frac{m}{k}$ is an irreducible fraction. Further we note that the algebras pB_q and $pB_{\tilde{q}^\xi}$ are isomorphic for

$$q = e^{i\frac{\pi}{2}\frac{m}{k}} \text{ and } \tilde{q}^\xi = e^{i\frac{\pi}{2}\frac{2k+\xi m}{k}} \quad (3.1)$$

whenever $\xi = +$ and $m = 1, 2, \dots, 2k - 1$ or $\xi = -$ and $m = 1, 2, \dots, k - 1$, since the generators $\tilde{a}^\pm = a^{\pm\xi}$ and $\tilde{K} = -\xi K$ of $pB_{\tilde{q}^\xi}$ satisfy the defining relations (2.1)-(2.3) for pB_q . The case $\xi = +$ indicates that we can set $m \in \{1, 2, \dots, 2k - 1\}$; the case $\xi = -$ further shows that without loss of generality we can assume that $m \in \{1, 2, \dots, k - 1\}$. The case $k = 1$ is excluded from these conditions. Thus, without losing any of the algebras pB_q for which the Fock space $F(p)$ is indecomposable, we restrict m and k to values, which we call admissible. The fraction $\frac{m}{k}$ is said to be admissible if

$$k = 2, 3, \dots \quad (3.2a)$$

$$\frac{m}{k} \in \left\{ \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k} \right\} \quad (3.2b)$$

$$\frac{m}{k} \text{ is an irreducible fraction (} m \text{ and } k \text{ are co-prime).} \quad (3.2c)$$

Passing to a discussion of the root of 1 representations, we first note that the vectors

$$|p; 0\rangle, |p; 2k\rangle, |p; 4k\rangle, \dots, |p; 2kN\rangle, \dots \quad (3.3)$$

are singular vectors in $F(p)$. Indeed $[2kN] = 0$ and therefore (see (2.9c))

$$a^- |p; 2kN\rangle = 0 \quad N \in \mathbf{Z}_+. \quad (3.4)$$

The subspaces

$$V_{|p; 2kN\rangle} = \text{span}\{|p; n\rangle | n \geq 2kN\} \quad (3.5)$$

are infinite-dimensional invariant subspaces of $F(p)$ with highest weight vectors $|p; 2kN\rangle$. Clearly

$$F(p) = V_{|p; 0\rangle} \supset V_{|p; 2k\rangle} \supset V_{|p; 4k\rangle} \supset \dots \supset V_{|p; 2kN\rangle} \supset \dots \quad (3.6)$$

For each $N < M \in \mathbf{Z}_+$ define a $2(M - N)k$ -dimensional factor space

$$W_{|p; 2kN\rangle, M} = V_{|p; 2kN\rangle} / V_{|p; 2kM\rangle}. \quad (3.7)$$

Let ξ_x be the equivalence class of $x \in \xi_x$. The vectors

$$\xi_{|p; 2kN\rangle}, \xi_{|p; 2kN+1\rangle}, \xi_{|p; 2kN+2\rangle}, \dots, \xi_{|p; 2kM-1\rangle} \quad (3.8)$$

constitute a basis in $W_{|p; 2kN\rangle, M}$. The relations

$$a^\pm \xi_{|p; n\rangle} = \xi_{a^\pm |p; n\rangle} \quad K \xi_{|p; n\rangle} = \xi_{K |p; n\rangle} \quad (3.9)$$

endow $W_{|p; 2kN\rangle, M}$ with a structure of a pB_q module. Observe that $a^+ \xi_{|p; 2kM-1\rangle} = 0$.

We shall simplify the notation, identifying $\xi_{|p;n\rangle}$ with its representative $|p;n\rangle$. Then, adding to eqs. (2.9) the condition $a^+|p;2kM-1\rangle=0$, one obtains the transformations of $W_{|p;2kN\rangle,M}$.

We summarise. The space $W_{|p;2kN\rangle,M}$, $N < M \in \mathbf{Z}_+$ is a $2(M-N)k$ -dimensional pB_q module with a highest weight $|p;2kN\rangle$, a basis

$$|p;n\rangle \quad n = 2kN, 2kN+1, \dots, 2kM-1 \quad (3.10)$$

and transformation relations

$$K|p;n\rangle = q^{2n+p}|p;n\rangle \quad (3.11a)$$

$$a^+|p;n\rangle = |p;n+1\rangle \quad n \neq 2kM-1 \quad (3.11b)$$

$$a^+|p;2kM-1\rangle = 0 \quad (3.11c)$$

$$a^-|p;n\rangle = [n]\{n+p-1\}|p;n-1\rangle \quad \text{for } n = \text{even number} \quad (3.11d)$$

$$a^-|p;n\rangle = [n+p-1]\{n\}|p;n-1\rangle \quad \text{for } n = \text{odd number.} \quad (3.11e)$$

If $M-N > 1$, eqs. (3.11) define an indecomposable finite-dimensional representation of pB_q in $W_{|p;2kN\rangle,M}$ for any $p \in \mathbf{C}$. If $M-N=1$, $W_{|p;2kN\rangle,N+1}$ is either irreducible or indecomposable. In this case the vectors

$$|p;2kN\rangle, |p;2kN+1\rangle, |p;2kN+2\rangle, \dots, |p;2k(N+1)-1\rangle \quad (3.12)$$

constitute a basis in $W_{|p;2kN\rangle,N+1}$.

Proposition 2. For each $s \in \mathbf{Z}$ and $N \in \mathbf{Z}_+$ the pB_q modules $W_{|p;0\rangle,1}$ and $W_{|p+4ks;2kN\rangle,N+1}$ are equivalent; they carry one and the same $2k$ -dimensional representation of pB_q .

Proof. Define a one-to-one map

$$\varphi(|p;n\rangle) = |p+4ks;n+2kN\rangle \quad n = 0, 1, 2, \dots, 2k-1 \quad (3.13)$$

of the basis in $W_{|p;0\rangle,1}$ onto the basis of $W_{|p+4ks;2kN\rangle,N+1}$ and extend it to a linear map from $W_{|p;0\rangle,1}$ onto $W_{|p+4ks;2kN\rangle,N+1}$. From (3.11) one derives that φ commutes with pB_q ,

$$a\varphi(|p;n\rangle) = \varphi(a|p;n\rangle) \quad a = a^\pm, K. \quad (3.14)$$

i.e., φ is an intertwining operator. In particular the matrices of the generators a^\pm and K are the same in the basis

$$|p;0\rangle, |p;1\rangle, |p;2\rangle, \dots, |p;2k-1\rangle \quad (3.15)$$

of $W_{|p;0\rangle,1}$ and in the basis $|p+4ks;2kN\rangle, |p+4ks;2kN+1\rangle, \dots, |p+4ks;2k(N+1)-1\rangle$ of $W_{|p+4ks;2kN\rangle,N+1}$, respectively. This completes the proof.

In view of Proposition 2 we shall consider from now on only the vacuum modules $W_{|p;0\rangle,1}$, restricting also the values of p to the interval

$$0 < \text{Re}(p) \leq 4k. \quad (3.16)$$

We often write $W(|p; m\rangle, |p; n\rangle)$ whenever we wish to indicate that

$$|p; m\rangle, |p; m+1\rangle, |p; m+2\rangle, \dots, |p; n\rangle \quad (3.17)$$

is a basis in the linear space $W(|p; m\rangle, |p; n\rangle)$. In view of (3.15)

$$W_{|p;0\rangle,1} = W(|p; 0\rangle, |p; 2k-1\rangle). \quad (3.18)$$

We proceed to study in detail the structure of the Fock spaces for different admissible values of $\frac{m}{k}$. To this end we will consider three cases: 3.1 k is even, m is odd; 3.2 k is odd, m is odd; 3.3 k is odd, m is even.

3.1 The case $k=\text{even}$, $m=\text{odd}$

If p is not an integer the coefficients $\{n+p-1\}$ in (2.9c) and $[n+p-1]$ in (2.9d) never vanish. Therefore the vectors (3.3) are the only singular vectors in $F(p)$ and the vacuum $|p; 0\rangle$ is the only singular vector in $W_{|p;0\rangle,1}$. The eigenvalues of K on $|p; 0\rangle$ are different for different p , obeying (3.16). This gives rise to the following

Proposition 3. The pB_q modules $W(|p; 0\rangle, |p; 2k-1\rangle)$ are simple for $p \notin \{1, 2, \dots, 4k\}$. All of them are $2k$ -dimensional. The irreps corresponding to different p from (3.16) are inequivalent.

The transformation of the basis (3.15) follows from eqs. (3.11) at $N = M-1 = 0$. The relations below are written in a slightly more general form in order to accomodate also other cases. For

$$|p; 0\rangle, |p; 1\rangle, \dots, |p; L\rangle \quad (3.19a)$$

set

$$K|p; n\rangle = q^{2n+p}|p; n\rangle \quad (3.19b)$$

$$a^-|p; n\rangle = [n]\{n+p-1\}|p; n-1\rangle \quad \text{for } n = \text{even number} \quad (3.19c)$$

$$a^-|p; n\rangle = [n+p-1]\{n\}|p; n-1\rangle \quad \text{for } n = \text{odd number.} \quad (3.19d)$$

$$a^+|p; n\rangle = |p; n+1\rangle \quad n \neq L \quad (3.19e)$$

$$a^+|p; L\rangle = 0. \quad (3.19f)$$

If $L = 2k-1$ the above equations give the transformation of $W(|p; 0\rangle, |p; 2k-1\rangle)$.

3.1a Representations with even p

All modules $W(|p; 0\rangle, |p; 2k-1\rangle)$ corresponding to even values of p are no more irreducible.

Proposition 4. To each $p \in \{2, 4, \dots, 4k\}$ there corresponds a simple pB_q module $W(|p; 0\rangle, |p; L\rangle)$ with a basis $|p; 0\rangle, |p; 1\rangle, \dots, |p; L\rangle$ and values of L as follows:

$$L = 2k - p \quad \text{for } p \in \{2, 4, \dots, 2k\} \quad (3.20)$$

$$L = 4k - p \quad \text{for } p \in \{2k+2, 2k+4, \dots, 4k\}. \quad (3.21)$$

The transformation of the basis is described with the eqs. (3.19) for the above values of L . All $2k$ modules carry different, inequivalent irreps of pB_q .

Proof. We consider in detail the case $p \in \{2, 4, \dots, 2k\}$. The module $W(|p; 0\rangle, |p; 2k-1\rangle)$ contains only two singular vectors $|p; 0\rangle$ and $|p; 2k-p+1\rangle$:

$$a^-|p; 0\rangle = 0 \quad a^-|p; 2k-p+1\rangle = 0. \quad (3.22)$$

In view of (3.19e) $W(|p; 0\rangle, |p; 2k-1\rangle)$ is indecomposable. Its invariant subspace $W(|p; 2k-p+1\rangle, |p; 2k-1\rangle)$ is simple. The factor space

$$W(|p; 0\rangle, |p; 2k-1\rangle)/W(|p; 2k-p+1\rangle, |p; 2k-1\rangle)$$

with a basis

$$\xi_{|p; 0\rangle}, \xi_{|p; 1\rangle}, \dots, \xi_{|p; 2k-p\rangle} \quad (3.23)$$

is turned into an irreducible pB_q module, setting

$$a\xi_{|p; n\rangle} = \xi_{a|p; n\rangle} \quad a = a^\pm, K. \quad (3.24)$$

Therefore

$$a^+\xi_{|p; 2k-p\rangle} = \xi_{|p; 2k-p+1\rangle} = 0. \quad (3.25)$$

As before we identify the equivalence classes with their representatives and in particular

$$\xi_{|p; n\rangle} = |p; n\rangle. \quad (3.26)$$

Then

$$W(|p; 0\rangle, |p; 2k-1\rangle)/W(|p; 2k-p+1\rangle, |p; 2k-1\rangle) = W(|p; 0\rangle, |p; 2k-p\rangle). \quad (3.27)$$

From (3.24) and (3.25) one derives the transformation relations of $W(|p; 0\rangle, |p; 2k-p\rangle)$, which are given with eqs. (3.19) for $L = 2k-p$. The transformations of the invariant subspaces are described also with (3.19), but for $n = 2k-p+1, 2k-p+2, \dots, 2k-1 = L$.

The cases with $p \in \{2k+2, 2k+4, \dots, 4k\}$ are similar. The only singular vectors in $W(|p; 0\rangle, |p; 2k-1\rangle)$ are the vacuum $|p; 0\rangle$ and $|p; 4k-p+1\rangle$. Therefore the invariant subspace $W(|p; 4k-p+1\rangle, |p; 2k-1\rangle)$ is irreducible and its transformations are described with (3.19) for $n = 4k-p+1, 4k-p+2, \dots, 2k-1 = L$. The factor space $W(|p; 0\rangle, |p; 4k-p\rangle)$ is also irreducible and transforms according to (3.19) with $L = 4k-p$.

So far we have four kinds of simple pB_q modules. We proceed to show that

$$\begin{aligned} W(|p; 2k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 4k-p'\rangle) \\ \text{for } p &= 4k-p'+2 \in \{2, 4, \dots, 2k\} \end{aligned} \quad (3.28)$$

$$\begin{aligned} W(|p; 4k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 2k-p'\rangle) \\ \text{for } p &= 4k-p'+2 \in \{2k+2, 2k+4, \dots, 4k\} \end{aligned} \quad (3.29)$$

where the equality means that the corresponding modules are equivalent, they carry equivalent representations of pB_q .

In order to prove (3.28) we set a one to one linear map

$$\varphi : W(|p; 2k - p + 1\rangle, |p; 2k - 1\rangle) \rightarrow W(|p'; 0\rangle, |p'; 4k - p'\rangle) \quad (3.30)$$

which restricted to the corresponding basisses read:

$$\varphi(|p; n\rangle) = |p'; n'\rangle, \quad n = 2k - p + 1, 2k - p + 2, \dots, 2k - 1 \iff n' = 0, 1, 2, \dots, 4k - p'. \quad (3.31)$$

It is straightforward to check, using eqs. (3.19), that φ commutes with the pB_q generators, i.e., it is an intertwining operator.

The proof of (3.29) is similar. It remains to show that all $2k$ modules (3.20) and (3.21), i.e.,

$$W(|p; 0\rangle, |p; 2k - p\rangle) \text{ and } W(|p + 2k; 0\rangle, |p + 2k; 2k - p\rangle) \quad p \in \{2, 4, \dots, 2k\} \quad (3.32)$$

are inequivalent. To this end we observe that the modules corresponding to different p in (3.32) have different dimensions. The modules with the same dimensions, namely $W(|p; 0\rangle, |p; 2k - p\rangle)$ and $W(|p + 2k; 0\rangle, |p + 2k; 2k - p\rangle)$ have different spectrum of the Cartan generator K and therefore are also inequivalent. This completes the proof.

It has been noted in [37, 43] that the Casimir operator of $U_q[osp(1/2)]$ is no longer sufficient to label the root of unity representations. In particular this is the case with the modules (3.32), which have the same dimension. The Casimir operator [36] reads in our notation:

$$\begin{aligned} 2C_2(q) &= q^2 K^2 + q^{-2} K^{-2} \\ &+ (q^2 - q^{-2})(q - q^{-1})(q^2 K + q^{-2} K^{-1}) a^- a^+ - (q^2 - q^{-2})^2 (a^-)^2 (a^+)^2. \end{aligned} \quad (3.33)$$

Its eigenvalue is one and the same,

$$C_2(q) = \frac{1}{2}(q^{2p-2} + q^{-2p+2}) + 2 \quad (3.34)$$

on four inequivalent modules, namely

$$\begin{aligned} W(|p; 0\rangle, |p; 2k - p\rangle) \quad W(|p + 2k; 0\rangle, |p + 2k; 2k - p\rangle) \quad dim = 2k - p + 1 \\ W(|2k - p + 2; 0\rangle, |2k - p + 2; p - 2\rangle) \quad W(|4k - p + 2; 0\rangle, |4k - p + 2; p - 2\rangle) \quad dim = p - 1. \end{aligned} \quad (3.35)$$

Observe that pairwise the dimensions of these modules are different.

Let us add that the additional central elements [37, 38] $(a^\pm)^{4k}$ and $(K)^{2k}$ do not distinguish among the inequivalent modules with the same dimension. In fact the operators $(a^\pm)^{4k}$ vanish within each simple module $W(|p; 0\rangle, |p; L\rangle)$.

3.1b Representations with odd p

Proposition 5. To each $p \in \{1, 3, \dots, 4k-1\}$ there corresponds an irreducible pB_q module $W(|p; 0\rangle, |p; L\rangle)$ with a basis $|p; 0\rangle, |p; 1\rangle, \dots, |p; L\rangle$ and values of L as follows:

$$L = k - p \quad \text{for } p \in \{1, 3, \dots, k-1\} \quad (3.36a)$$

$$L = 3k - p \quad \text{for } p \in \{k+3, k+5, \dots, 3k-1\} \quad (3.36b)$$

$$L = 5k - p \quad \text{for } p \in \{3k+3, 3k+5, \dots, 4k-1\} \quad (3.36c)$$

$$L = 2k - 1 \quad \text{for } p = k+1, 3k+1. \quad (3.36d)$$

The transformations of the basis are given with the eqs. (3.19) for the corresponding values of L , indicated above. All these $2k$ modules carry inequivalent irreps of pB_q .

Proof. The proof is similar to the one of Proposition 4. We stress on certain points only. In this case we have both simple and indecomposable modules $W(|p; 0\rangle, |p; 2k-1\rangle)$. More precisely,

$$W(|p; 0\rangle, |p; 2k-1\rangle) \quad p = k+1, 3k+1 \quad (3.37)$$

are irreducible; the rest of the modules

$$W(|p; 0\rangle, |p; 2k-1\rangle) \quad p \in \{1, 3, 5, \dots, 4k-1\} \quad p \neq k+1, 3k+1 \quad (3.38)$$

are indecomposable. Each module in (3.38) contains apart from the vacuum only one more singular vector. Therefore the invariant subspaces and the factor spaces are irreducible. Also here each invariant subspace is equivalent to a factor space. More precisely,

$$\begin{aligned} W(|p; k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 3k-p'\rangle) \\ \text{for } p &= 2k-p'+2 \in \{1, 3, \dots, k-1\} \end{aligned} \quad (3.39)$$

$$\begin{aligned} W(|p; 3k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; k-p'\rangle) \\ \text{for } p &= 2k-p'+2 \in \{k+3, k+5, \dots, 2k+1\} \end{aligned} \quad (3.40)$$

$$\begin{aligned} W(|p; 3k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 5k-p'\rangle) \\ \text{for } p &= 6k-p'+2 \in \{2k+3, 2k+5, \dots, 3k-1\} \end{aligned} \quad (3.41)$$

$$\begin{aligned} W(|p; 5k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 3k-p'\rangle) \\ \text{for } p &= 6k-p'+2 \in \{3k+3, 3k+5, \dots, 4k-1\}. \end{aligned} \quad (3.42)$$

Therefore (up to equivalence) we are left only with the vacuum modules (3.36). Among them one finds all of the time pairs with the same dimension, namely

$$\begin{aligned} \dim[W(|p; 0\rangle, |p; k-p\rangle)] &= \dim[W(|p+2k; 0\rangle, |p+2k; k-p\rangle)] = k-p+1 \\ \text{for } p &\in \{1, 3, \dots, k-1\} \end{aligned} \quad (3.43)$$

$$\begin{aligned} \dim[W(|p; 0\rangle, |p; 3k - p\rangle)] &= \dim[W(|p + 2k; 0\rangle, |p + 2k; 3k - p\rangle)] = 3k - p + 1 \\ \text{for } p &\in \{k + 3, k + 5, \dots, 2k - 1\}. \end{aligned} \quad (3.44)$$

Also in this case the modules with the same dimension cannot be separated by the Casimir operator. They have however different spectrum of the Cartan generator K . Hence they carry inequivalent representations of pB_q .

3.2 The case $k=\text{odd}$, $m=\text{odd}$

For any $N \in \mathbf{Z}_+$ (see (2.9)) $a^-|p; kN\rangle = 0$. Therefore the Fock spaces contain more singular vectors than in the general case (see (3.3)). Now

$$|p; 0\rangle, |p; k\rangle, |p; 2k\rangle, \dots, |p; kN\rangle, \dots \quad (3.45)$$

are singular vectors in $F(p)$. For each N the subspace

$$V_{|p; kN\rangle} = \text{span}\{|p; n\rangle | n \geq kN\} \quad (3.46)$$

is an infinite-dimensional indecomposable invariant subspace of $F(p)$. Because of the inclusions

$$F(p) = V_{|p; 0\rangle} \supset V_{|p; k\rangle} \supset V_{|p; 2k\rangle} \supset \dots \supset V_{|p; kN\rangle} \supset \dots$$

one can build up various indecomposable finite-dimensional pB_q modules. For each $N < M \in \mathbf{Z}_+$

$$W(|p; kN\rangle, |p; kM - 1\rangle) = V_{|p; kN\rangle} / V_{|p; kM\rangle} \quad (3.47)$$

is an $(M - N)k$ -dimensional indecomposable pB_q module with singular vectors $|p; kN\rangle, |p; (k + 1)N\rangle, \dots, |p; k(M - 1)\rangle$. As before we do not distinguish between the equivalence classes and their representatives. Since

$$a^+ \xi_{|p; kM - 1\rangle} = \xi_{a^+ |p; kM - 1\rangle} = 0 \quad \Leftrightarrow \quad a^+ |p; kM - 1\rangle = 0 \quad (3.48)$$

the transformation of the factor space (3.47) is given with eqs. (3.11), where one has to replace $2kM - 1$ by $kM - 1$ in (3.11b,c). If $M - N = 1$ $W(|p; kN\rangle, |p; k(N + 1) - 1\rangle) = V_{|p; kN\rangle} / V_{|p; k(N + 1)\rangle}$ is either irreducible or indecomposable. We are mainly concerned with the classification of the irreducible pB_q modules. Therefore, as a first step, we identify some equivalent modules. From Proposition 2 we know that for a given pB_q algebra (i.e. for a fixed admissible fraction $\frac{m}{k}$) all irreps can be extracted from the collection of the modules

$$W(|p; 0\rangle, |p; 2k - 1\rangle) \quad 0 < \text{Re}(p) \leq 4k. \quad (3.49)$$

According to (3.47) (when $N = 0$, $M = 2$) the above module is indecomposable. It contains at least two singular vectors, namely $|p; 0\rangle$ and $|p; k\rangle$. The subspace $W(|p; k\rangle, |p; 2k - 1\rangle)$ is an invariant pB_q subspace and therefore the factor space

$$W(|p; 0\rangle, |p; k - 1\rangle) = W(|p; 0\rangle, |p; 2k - 1\rangle) / W(|p; k\rangle, |p; 2k - 1\rangle) \quad (3.50)$$

is also a pB_q module.

Proposition 6. The collection of all pB_q modules $W(|p; 0\rangle, |p; k-1\rangle)$ is equivalent to the collection of all invariant subspaces $W(|p; k\rangle, |p; 2k-1\rangle)$ when $0 < \text{Re}(p) \leq 4k$. More precisely,

$$W(|p; 0\rangle, |p; k-1\rangle) = W(|p+2k; k\rangle, |p+2k; 2k-1\rangle) \quad 0 < \text{Re}(p) \leq 2k \quad (3.51a)$$

$$W(|p; 0\rangle, |p; k-1\rangle) = W(|p-2k; k\rangle, |p-2k; 2k-1\rangle) \quad 2k < \text{Re}(p) \leq 4k. \quad (3.51b)$$

Proof. The transformations of $W(|p; 0\rangle, |p; k-1\rangle)$ are given with eqs. (3.19) for $L = k-1$. The same equations describe the action of the pB_q generators on $W(|p; k\rangle, |p; 2k-1\rangle)$ for $L = 2k-1$ and $n = k, k+1, \dots, 2k-1$. The corresponding intertwining operator φ , defined on the basisses, read:

$$\varphi(|p; n\rangle) = |p+2\xi k; n+k\rangle \quad n = 0, 1, 2, \dots, k-1 \quad (3.52)$$

where $\xi = 1$ corresponds to (3.51a) and $\xi = -1$ — to (3.51b).

In view of Proposition 6 we shall consider only the vacuum modules $W(|p; 0\rangle, |p; k-1\rangle)$ restricting as before the values of p to the interval (3.16).

If p is not an integer the coefficients $\{n+p-1\}$ in (2.9c) and $[n+p-1]$ in (2.9d) never vanish. Therefore the vectors (3.45) are the only singular vectors in $F(p)$ and the vacuum $|p; 0\rangle$ is the only singular vector in $W(|p; 0\rangle, |p; k-1\rangle)$. We have obtained the following result.

Proposition 7. The pB_q modules $W(|p; 0\rangle, |p; k-1\rangle)$ are simple for $p \notin \{1, 2, \dots, 4k\}$. All of them are k -dimensional. The irreps corresponding to different p from (3.16) are inequivalent. The transformation of the basis

$$|p; 0\rangle, |p; 1\rangle, \dots, |p; k-1\rangle \quad (3.53)$$

is given with eqs. (3.19) for $L=k-1$.

3.2a Representations with even p

Proposition 8. To each $p \in \{2, 4, \dots, 4k\}$ there corresponds an irreducible pB_q module $W(|p; 0\rangle, |p; L\rangle)$ with a basis $|p; 0\rangle, |p; 1\rangle, \dots, |p; L\rangle$ and values of L as follows:

$$L = k - p \quad \text{for } p \in \{2, 4, \dots, k-1\} \quad (3.54a)$$

$$L = 2k - p \quad \text{for } p \in \{k+3, k+5, \dots, 2k\} \quad (3.54b)$$

$$L = 3k - p \quad \text{for } p \in \{2k+2, 2k+4, \dots, 3k-1\} \quad (3.54c)$$

$$L = 4k - p \quad \text{for } p \in \{3k+3, 3k+5, \dots, 4k\} \quad (3.54d)$$

$$L = k - 1 \quad \text{for } p = k+1, 3k+1. \quad (3.54e)$$

The transformations of the basis are given with the eqs. (3.19) for the corresponding values of L in (3.54). All $2k$ modules (3.54) carry inequivalent irreps of pB_q .

Proof. We sketch the proof. The modules

$$W(|p; 0\rangle, |p; k-1\rangle) \quad \text{with } p = k+1, 3k+1 \quad (3.55)$$

remain irreducible, whereas

$$W(|p; 0\rangle, |p; k-1\rangle) \quad p \in \{2, 4, \dots, 4k\} \quad p \neq k+1, 3k+1 \quad (3.56)$$

are indecomposable. Each module in (3.56) contains apart from the vacuum only one more singular vector. Therefore the invariant subspaces and the factor spaces are irreducible. Each invariant subspace is equivalent to a factor space:

$$\begin{aligned} W(|p; k-p+1\rangle, |p; k-1\rangle) &= W(|p'; 0\rangle, |p'; 2k-p'\rangle) \\ \text{for } p &= 2k-p'+2 \in \{2, 4, \dots, k-1\} \end{aligned} \quad (3.57)$$

$$\begin{aligned} W(|p; 2k-p+1\rangle, |p; k-1\rangle) &= W(|p'; 0\rangle, |p'; 3k-p'\rangle) \\ \text{for } p &= 4k-p'+2 \in \{k+3, k+5, \dots, 2k\} \end{aligned} \quad (3.58)$$

$$\begin{aligned} W(|p; 3k-p+1\rangle, |p; k-1\rangle) &= W(|p'; 0\rangle, |p'; 4k-p'\rangle) \\ \text{for } p &= 6k-p'+2 \in \{2k+2, 2k+4, \dots, 3k-1\} \end{aligned} \quad (3.59)$$

$$\begin{aligned} W(|p; 4k-p+1\rangle, |p; k-1\rangle) &= W(|p'; 0\rangle, |p'; k-p'\rangle) \\ \text{for } p &= 4k-p'+2 \in \{3k+3, 3k+5, \dots, 4k\}. \end{aligned} \quad (3.60)$$

Thus, (up to equivalence) one can consider only the vacuum modules (3.54). Some of them have one and the same dimension, namely

$$\begin{aligned} \dim[W(|p; 0\rangle, |p; k-p\rangle)] &= \dim[W(|p+2k; 0\rangle, |p+2k; k-p\rangle)] = k-p+1 \\ \text{for } p &\in \{2, 4, \dots, k-1\} \end{aligned} \quad (3.61)$$

$$\begin{aligned} \dim[W(|p; 0\rangle, |p; 2k-p\rangle)] &= \dim[W(|p+2k; 0\rangle, |p+2k; 2k-p\rangle)] = 2k-p+1 \\ \text{for } p &\in \{k+3, k+5, \dots, 2k\} \end{aligned} \quad (3.62)$$

$$\dim[W(|k+1; 0\rangle, |k+1; k-1\rangle)] = \dim[W(|3k+1; 0\rangle, |3k+1; k-1\rangle)] = k. \quad (3.63)$$

The modules with the same dimension cannot be separated by the Casimir operator. The spectrum of K is however different. Therefore these modules are inequivalent.

3.2b Representations with odd p

This collection of representations is somewhat different from the other cases. All Fock modules $W(|p; 0\rangle, |p; k-1\rangle)$ remain irreducible for $p \in \{1, 3, 5, \dots, 4k-1\}$.

3.3 The case $k=\text{odd}$, $m=\text{even}$

According to *Proposition 2* the irreps (up to equivalence) are realized in the vacuum modules $W(|p; 0\rangle, |p; 2k-1\rangle)$ with $0 < \text{Re}(p) \leq 4k$. For the algebras from this class one can further restrict the values of p .

Proposition 9. The following modules are equivalent:

$$W(|p; 0\rangle, |p; 2k-1\rangle) = W(|p+2k; 0\rangle, |p+2k; 2k-1\rangle) \quad \text{for } 0 < \text{Re}(p) \leq 2k \quad (3.64)$$

$$W(|p; 0\rangle, |p; 2k-1\rangle) = W(|p+k; 0\rangle, |p+k; 2k-1\rangle) \quad \text{for } 0 < \text{Re}(p) \leq k \text{ and } m = 4(\text{mod } 4). \quad (3.65)$$

Proof. The intertwining operators, written on the basisses read

$$\varphi(|p; n\rangle) = |p + \alpha k; n\rangle \quad n = 0, 1, \dots, 2k-1 \quad (3.66)$$

where $\alpha = 2$ for (3.64) and $\alpha = 1$ for (3.65).

Hence, without loss of generality we assume

$$0 < \text{Re}(p) \leq 2k \quad \text{if } m = 2(\text{mod } 4) \quad (3.67)$$

$$0 < \text{Re}(p) \leq k \quad \text{if } m = 4(\text{mod } 4). \quad (3.68)$$

Proposition 10. The pB_q module $W(|p; 0\rangle, |p; 2k-1\rangle)$ is irreducible if p is not an integer. All such modules are $2k$ -dimensional. The irreps corresponding to different p from (3.67) and (3.68) are inequivalent. The transformations of the basis (3.15) is described with eqs. (3.19) for $L = 2k-1$.

Proof. The same as in Proposition 3.

If p is an integer each space $W(|p; 0\rangle, |p; 2k-1\rangle)$ is indecomposable. Apart from $|p; 0\rangle$ it contains only one more singular vector $|p; L+1\rangle$, where

$$L = 2k - p \quad \text{for } p = \text{even} \quad (3.69a)$$

$$L = k - p \quad \text{for } p \in \{1, 3, \dots, k\} \quad (3.69b)$$

$$L = 3k - p \quad \text{for } p \in \{k+2, k+4, \dots, 2k-1\} \quad (m = 2(\text{mod } 4)). \quad (3.69c)$$

In all cases $W(|p; L+1\rangle, |p; 2k-1\rangle)$ is an irreducible pB_q module. So is the corresponding factor space

$$W(|p; 0\rangle, |p; L\rangle) = W(|p; 0\rangle, |p; 2k-1\rangle) / W(|p; L+1\rangle, |p; 2k-1\rangle). \quad (3.70)$$

Each invariant subspace is equivalent to a certain factor space:

$$W(|1; k\rangle, |1; 2k-1\rangle) = W(|1; 0\rangle, |1; k-1\rangle) \quad (3.71)$$

$$\begin{aligned} W(|p; k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 3k-p'\rangle) \\ \text{for } m &= 2(\text{mod } 4), \quad p = 2k - p' + 2 \in \{3, 5, \dots, k\} \end{aligned} \quad (3.72)$$

$$\begin{aligned} W(|p; 3k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; k-p'\rangle) \\ \text{for } m &= 2(\text{mod } 4), \quad p = 2k - p' + 2 \in \{k+2, k+4, \dots, 2k-1\} \end{aligned} \quad (3.73)$$

$$\begin{aligned} W(|p; k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 2k-p'\rangle) \\ \text{for } m &= 4(\text{mod } 4), \quad p = k - p' + 2 \in \{3, 5, \dots, k\} \end{aligned} \quad (3.74)$$

$$\begin{aligned} W(|p; 2k-p+1\rangle, |p; 2k-1\rangle) &= W(|p'; 0\rangle, |p'; 2k-p'\rangle) \\ \text{for } p &= 2k - p' + 2 = \text{even} \end{aligned} \quad (3.75)$$

We skip the explicit form of the intertwining operators and collect the result in a proposition.

Proposition 11. To each integer p , $0 < p \leq 2k$ if $m = 2(\text{mod } 4)$, and $0 < p \leq k$ if $m = 4(\text{mod } 4)$, there corresponds an irreducible pB_q module $W(|p; 0\rangle, |p; L\rangle)$ with a basis $|p; 0\rangle, |p; 1\rangle, \dots, |p; L\rangle$. All such modules are inequivalent. Their transformations under the action of pB_q are given with eqs. (3.19) for the corresponding values of L in (3.69).

4. Unitarizable representations

In the present Section we classify the unitarizable Fock representations of the deformed para-Bose superalgebra pB_q . The concept of an unitarizable representation of an arbitrary associative algebra A depends on the definition of the antilinear antiinvolution $\omega : A \rightarrow A$ and on the metric in the corresponding A -module. One and the same representation can be unitarizable with respect to one antiinvolution and not unitarizable with respect to another one.

Having in mind the physical condition (1.1) and the requirement the "Hamiltonian" H to be Hermitian operator, we define ω on the generators a^\pm and K as

$$\omega(a^\pm) = a^\mp \quad \omega(K^{\pm 1}) = K^{\mp 1} \quad (4.1)$$

and extend it on pB_q as an antilinear antiinvolution, namely

$$\omega(\alpha a + \beta b) = \alpha^* \omega(a) + \beta^* \omega(b) \quad \omega(ab) = \omega(b) \omega(a) \quad (4.2)$$

for all $a, b \in pB_q$ and $\alpha, \beta \in \mathbf{C}$ with α^* being the complex conjugate of α .

The representation of pB_q in a Hilbert space W with scalar product $(\ , \)$ is unitarizable if

$$(ax, y) = (x, \omega(a)y) \quad \forall a \in pB_q \text{ and } x, y \in W \quad (4.3)$$

or, equivalently, if $\omega(a) = a^\dagger$. On the generators of pB_q the latter condition reads:

$$(a^-)^\dagger = a^+ \quad K^\dagger = K^{-1}. \quad (4.4)$$

The problem is to select those irreducible modules $W(|p; 0\rangle, |p; L\rangle)$ for which the unitarity condition (4.4) can be satisfied. To this end we introduce a new basis

$$|p; n\rangle = \alpha(p; n)|p; n\rangle \quad n = 1, 2, \dots, L \quad \alpha(p; n) \in \mathbf{C} \quad (4.5)$$

which is declared to be orthonormed. Then the unitarity condition is equivalent to the requirement the following two conditions to be satisfied:

$$\left| \frac{\alpha(p; n)}{\alpha(p; n+1)} \right|^2 = \frac{2\sin(\frac{\pi}{2} \frac{m}{k}(n+p))\cos(\frac{\pi}{2} \frac{m}{k}(n+1))}{\sin(\frac{\pi m}{k})} \quad n = \text{even} \quad (4.6a)$$

$$\left| \frac{\alpha(p; n)}{\alpha(p; n+1)} \right|^2 = \frac{2\sin(\frac{\pi}{2} \frac{m}{k}(n+1))\cos(\frac{\pi}{2} \frac{m}{k}(n+p))}{\sin(\frac{\pi m}{k})} \quad n = \text{odd}. \quad (4.6b)$$

The unknowns in the above equations are the algebras pB_q , i.e., the admissible pairs $\frac{m}{k}$ and the irreducible modules $W(|p; 0\rangle, |p; L\rangle)$, i.e., the values of p and L . The eqs. (4.6) have solutions only if the r.h.s. of them is nonnegative number for any n from the basis in $W(|p; 0\rangle, |p; L\rangle)$. Thus, the problem is to solve a set of inequalities. Bellow we list the algebras pB_q with $q = e^{i\frac{\pi}{2}\frac{m}{k}}$ in terms of the admissible m and k and their unitarizable representations.

The algebra pB_q	Unitarizable modules	
(1) $m = 1, k = 3, 5, 7, \dots$	$W(p; 0\rangle, p; k-1\rangle)$	$0 \leq p \leq 2$
(2) $m = 1, k = 2, 4, 6, \dots$	$W(p; 0\rangle, p; k-p\rangle)$	$p = 1, 3, 5, \dots, k-1$
(3) $m = 1, k = 3, 5, 7, \dots$	$W(p; 0\rangle, p; k-p\rangle)$	$p = 2, 4, 6, \dots, k-1$
(4) $m = 1(mod 4), k = 2, 3, 4, \dots$	$W(k-1; 0\rangle, k-1; 1\rangle)$	
(5) $m = 3(mod 4), k = 2, 3, 4, \dots$	$W(3k-1; 0\rangle, 3k-1; 1\rangle)$	
(6) $m = 3, k = 10, 12, 14, \dots$	$W(3k-3; 0\rangle, 3k-3; 3\rangle)$	

We underline that everywhere in eqs. (4.7) only the admissible values of m and k are to be considered (see (3.2)).

The above equations indicate that the algebras pB_q corresponding to $m = 1$ and any odd k have a continuous class of unitarizable representations. In all other cases the number of the unitarizable irreps is finite and in fact each algebra with $m \neq 1$ has no more than two representation. In the cases (4) and (5) the representation is 2-dimensional. In the new basis (4.5) the matrices of the generators read:

$$a^- = \begin{pmatrix} 0 & \sqrt{\frac{\cos \frac{\pi}{2}\frac{m}{k}}{\sin \frac{\pi}{2}\frac{m}{k}}} \\ 0 & 0 \end{pmatrix} \quad a^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{\cos \frac{\pi}{2}\frac{m}{k}}{\sin \frac{\pi}{2}\frac{m}{k}}} & 0 \end{pmatrix} \quad K = \begin{pmatrix} ie^{-i\frac{\pi}{2}\frac{m}{k}} & 0 \\ 0 & ie^{i\frac{\pi}{2}\frac{m}{k}} \end{pmatrix} \quad (4.8)$$

Similarly, the 4-dimensional representation from the case (6) reads:

$$a^- = \begin{pmatrix} 0 & \sqrt{\frac{\cos \frac{9\pi}{2k}}{\sin \frac{3\pi}{2k}}} & 0 & 0 \\ 0 & 0 & \sqrt{2\sin \frac{3\pi}{k}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{\cos \frac{9\pi}{2k}}{\sin \frac{3\pi}{2k}}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K = \begin{pmatrix} ie^{-\frac{9i\pi}{2k}} & 0 & 0 & 0 \\ 0 & ie^{-\frac{3i\pi}{2k}} & 0 & 0 \\ 0 & 0 & ie^{\frac{3i\pi}{2k}} & 0 \\ 0 & 0 & 0 & ie^{\frac{9i\pi}{2k}} \end{pmatrix} \quad (4.9)$$

with a^+ represented by the transposed matrix of a^- . Note that the algebras with $m = 3$ and $k = 10, 12, 14, \dots$ have only two unitarizable irreps, namely (4.8) and (4.9); the algebras with $m = 3$ and $k = 2, 4, 6, 8$ and those with $m = 5, 7, 9, \dots$ have only a 2-dimensional unitarizable irrep. The algebras pB_q with even m have no unitarizable representations at all.

The transformation relations of all unitarizable modules can be written in a compact form. In the orthonormed basis $|p; n\rangle$ eqs. (3.19) read:

$$K|p; n\rangle = e^{i\frac{\pi}{2}\frac{m}{k}(2n+p)}|p; n\rangle \quad (4.10a)$$

$$a^-|p; n\rangle = \sqrt{\frac{2\sin(\frac{\pi}{2}\frac{m}{k}n)\cos(\frac{\pi}{2}\frac{m}{k}(p+n-1))}{\sin(\frac{\pi m}{k})}}|p; n-1\rangle \quad n = \text{even} \quad (4.10b)$$

$$a^-|p; n\rangle = \sqrt{\frac{2\sin(\frac{\pi}{2}\frac{m}{k}(p+n-1))\cos(\frac{\pi}{2}\frac{m}{k}n)}{\sin(\frac{\pi m}{k})}}|p; n-1\rangle \quad n = \text{odd} \quad (4.10c)$$

$$a^+|p; n\rangle = \sqrt{\frac{2\sin(\frac{\pi}{2}\frac{m}{k}(p+n))\cos(\frac{\pi}{2}\frac{m}{k}(n+1))}{\sin(\frac{\pi m}{k})}}|p; n+1\rangle \quad n = \text{even} \quad (4.10d)$$

$$a^+|p; n\rangle = \sqrt{\frac{2\sin(\frac{\pi}{2}\frac{m}{k}(n+1))\cos(\frac{\pi}{2}\frac{m}{k}(p+n))}{\sin(\frac{\pi m}{k})}}|p; n+1\rangle \quad n = \text{odd} \quad (4.10e)$$

We have skipped eq. (3.19f), since it is automatically satisfied: the creation operator a^+ , which is the negative root vector, annihilates the lowest weight vectors $|p; L\rangle$ within each unitarizable module.

5. Concluding remarks and discussions

We have studied root of unity representations of the deformed para-Bose algebra $pB_q = U_q[osp(1/2)]$ with a particular emphasis on the unitarizable irreps. All of them are realized in finite-dimensional modules with a highest and a lowest weight. The irreps from Sec. 3.2 and also all irreps, corresponding to integer p (except $p = k + 1, 3k + 1$ in Sec. 3.1b) are new.

In the nondeformed case the representations of the para-Bose operators, corresponding to an order of the statistics $p = 1$ reduce to usual Bose operators [42]. In [1] it was shown that similar relation holds in the deformed case for generic q . It is straightforward to check that in the cases $p = 1, m = 1, k = 2, 3, \dots$ eqs. (4.10) recover also all root of unity unitarizable irreps of the deformed Bose operators [24-27] as given in [1].

Using the approach of the present paper one can try to construct representations (including root of 1 representations) for $pB_q(n) = U_q[osp(1/2n)]$. To this end one can use n -pairs of deformed pB operators as given in [2,4,5]. The solution, however, is not going to be easy for arbitrary values of p , if one takes into account that the problem has not been solved even in the nondeformed case. Only the case with $p = 1$ is easy. It leads directly to root of 1 representations of $U_q[osp(1/2n)]$, if one uses q -commuting deformed Bose operators as defined in [5]. Other root of 1 representations based on a realization with commuting q -Bose operators (which means also the case $p = 1$) were obtained in [44]. In this relation we note that n pairs of commuting deformed Bose operators are already generators of $U_q[osp(1/2n)]$ (in the q -Bose representation). Therefore they provide the simplest q -Boson realization of $U_q[osp(1/2n)]$ [28].

Finally we mention that all our representations correspond to q being even root of unity: $q^{4k} = 1$. In case of deformed simple Lie algebras this seems to be the more difficult case. Complete results exist for q being only odd roots of 1 [45].

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